THE PSYCHOLOGY OF MATHEMATICS FOR INSTRUCTION

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To Jeremy, Paul, Daniel, and Jesse,
who let us watch while they learned.

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When we think back on our own school days, to the hours we spent on arithmetic, many of us remember laboring over pages of problems. Often these were pages of identical calculations, where only the numbers were varied. Or we worked with flash cards until we could shout out the answers immediately and with no mistakes. This kind of work was called ''drill and practice.'' It was supposed to help us achieve perfect mastery of basic addition, subtraction, multiplication, and division. It was to ensure that we would forever remember how to perform the arithmetic operations we had been taught.

Drill and practice has a place of long standing in the history of mathematics teaching, especially in arithmetic. At one time it was the major means of instruction. Today it is still part of the mathematics curriculum, although usually accompanied by concrete experiences or explanations of underlying mathematical principles. Most everyone accepts some form of practice as necessary. The reason, according to educators and lay people alike, is that ''practice makes perfect.'' Along with drill and practice come increases in speed and accuracy, which are two widely accepted criteria of computational proficiency. If children can execute calculations speedily and accurately, most people are satisfied that they ''know'' their computational skills.

What do psychologists know about the role of drill in establishing and maintaining computational proficiency? This chapter explores the historical and theoretical bases for including drill and practice in the mathematics curriculum. We begin with a look at a psychological theory—associationism—that provides one theoretical justification for the use of drill exercises. We have chosen to focus on E. L. Thorndike, who is, in a sense, the ''founding father'' of the psychology of mathematics instruction. As a psychologist Thorndike was firmly

rooted in a tradition of laboratory experimentalism; but he was also strongly committed to the task of translating laboratory findings into guidelines for classroom instruction. We also present the arguments advanced against drill methods by another psychologist, William Brownell. Because of its prominence in arithmetic teaching, practice has received a great deal of study, especially during the first half of this century, most of it geared to make drill better organized and more effective. We survey a line of research that attempted to determine the relative ease or difficulty of arithmetic problems—and to account for those differences—so that teachers could plan the proper amounts and sequences of practice. We describe a computer-assisted drill program as an example of one way psychologists have attempted to optimize amounts of practice and rates of progression through drill material. Finally, we present theory and research that indicate why it might be important to develop speed and accuracy in certain kinds of computations.

EDWARD L. THORNDIKE AND THE FORMATION OF BONDS

In 1922 a small book appeared, called *The Psychology of Arithmetic*. It was written by Edward L. Thorndike, a psychologist working at Teachers College of Columbia University, who helped to develop some early principles of stimulus-response learning psychology. Thorndike is perhaps best known in psychology for his statement of the *law of effect*, an early version of what we now call *principles of reinforcement*. He discovered this law, not in the context of a complex subject like mathematics but in the context of simple laboratory experiments with cats, dogs, monkeys, and chickens.

According to Thorndike, in any given situation an animal had a number of possible responses, and the action that would be performed depended on the strength of the "connection" or "bond" between the situation and the specific action. The experiment most frequently associated with this idea involved placing a cat in a wooden puzzle box that could be opened by tripping a latch. Naturally, the cat would object to being confined in such close quarters and would claw and scratch at the side of the box to get out. Eventually it would accidentally trip the latch, opening the door and escaping. Replaced in the box, the cat would again claw and scratch; but each time the experiment was repeated, the cat took less time to find its way out. Of all the clawing and scratching responses, only the one that opened the door was rewarded by the opportunity for escape. In Thorndike's conception, the cat was not "figuring out" how to open the box; rather the reward of escape was serving to strengthen the bonds between the experimental situation and the particular response that permitted escape. Hence Thorndike's formulation of the law of effect (Thorndike, 1913): "When a modifiable connection between a situation and a response is made and is accompanied or followed by a satisfying state of affairs, that connection's strength is increased: When made and accompanied or followed by an annoying state of affairs, its strength is decreased [p. 4]."

Though he experimented mostly with animals, Thorndike thought his learning principles should apply equally to humans. Along with many other psychologists of the time—called "connectionists" or "associationists"—Thorndike argued that all human behavior, thought as well as action, could be analyzed in terms of two simple constructs. When broken down into its most basic units, behavior would be found to consist of *stimuli,* or events external to the person, and *responses,* or things that people did in reaction to those external events. When a certain response was given to a certain stimulus and followed by a reward, then a bond, or association, began to be formed between the stimulus and the response. The more frequently a certain stimulus-response pair was rewarded, the stronger the bond. Thus the law of effect—a special case of the laws of association—suggested that practice followed by reward was an important way in which human learning took place.

Associations between stimuli and responses, bonds, and the law of effect—how could these principles, developed largely by observing animals perform the simplest of behaviors, be applied to something as complex as school learning? That was the question that Thorndike (1922) addressed in *The Psychology of Arithmetic*. The answer seemed straightforward because associationism held that all knowledge, even the most complex, was built of these simple connections. Learning thus consisted of establishing and strengthening the needed associations. "The aims of elementary education," Thorndike said, "when fully defined, will be found to be the production of changes in human nature represented by an almost countless list of connections or bonds whereby the pupil thinks or feels or acts in certain ways in response to the situations the school has organized and is influenced to think and feel and act similarly to similar situations when life outside of school confronts him with them [p. xi]."

Rather than simply announcing the laws of learning to teachers and educators, Thorndike set out to demonstrate how they could be applied to the problems of instruction. What teachers needed, he believed, was to find and make explicit the particular set of bonds that constituted arithmetic. Once well-organized lists of all these bonds could be drawn up, then rewarded practice would enable the law of effect to strengthen these bonds, and one could expect improved performance in arithmetic. Thorndike's book was an attempt to explain how the subject matter of arithmetic could be translated into psychologically formulated stimulus-response bonds.

Because children of elementary school age were not yet able to deduce the rules of arithmetic from examples and other rules, Thorndike reasoned, the task of instruction was to form carefully the necessary bonds and habits that would allow them to perform computations and solve problems. As a first step, one would have to select the bonds to be formed. Naturally, any carefully constructed

arithmetic curriculum, with or without benefit of psychological analysis, would divide the subject matter up into broadly defined topics. For example, multiplication would be treated as a composite of abilities, such as: "knowledge of multiplication tables up to 9 × 9; ability to multiply two (or more) place numbers when carrying is not required and no zeros occur in the multiplicand; ability to multiply by 2, 3, . . . , 9, with carrying;" and so forth, up to the ability to multiply two-place decimals (as with United States money), with fractions, and with mixed numbers. What Thorndike, as a psychologist, proposed was to analyze these abilities further into a detailed set of mental habits or connections, each of which would become a candidate for formation and strengthening. Figure 2.1 shows an analysis of simple addition in columns, of which Thorndike (1922) says: "The majority of teachers probably treat this as a simple application of the knowledge of the additions to 9 + 9, plus understanding of 'carrying.' On the contrary there are at least seven processes or minor functions involved in two-place column addition, each of which is psychologically distinct and requires distinct educational treatment [p. 52]."

Once the proper bonds were selected, how could they be formed and strengthened? This was where drill and practice came in. Proper drill and practice, according to Thorndike, involved presenting bonds in a carefully programmed way so that important bonds were practiced often, and lesser bonds, less often. So-called "propaedeutic" bonds, used only to facilitate learning new concepts, would be practiced temporarily but later drop out from disuse. For example, to add four 5's in a column, a child might be taught a propaedeutic bond like counting 5, 10, 15, 20; however, because this was to be replaced later

Learning to keep one's place in the column as one adds.
Learning to keep in mind the result of each addition until the next number is added to it.
Learning to add a seen to a thought-of number.
Learning to neglect an empty space in the columns.
Learning to neglect 0s in the columns.
Learning the application of the combinations to higher decades may for the less gifted pupils involve as much time and labor as learning all the original addition tables. And even for the most gifted child the formation of the connections '8 and 7 = 15' probably never quite insures the presence of the connections '38 and 7 = 45' and '18 + 7 = 25.'
Learning to write the figure signifying units rather than the total sum of a column. In particular, learning to write 0 in the cases where the sum of the column is 10, 20, etc. Learning to 'carry' also involves in itself at least two distinct processes, by whatever way it is taught.

FIG. 2.1 Thorndike's analysis of column addition into bonds. (From Thorndike, 1922.)

by the bond "four 5's are 20," it would receive only minimal practice. Bonds were recognized to have an effect on each other; hence Thorndike (1922) noted: "Every bond formed should be formed with due consideration of every other bond that has been or will be formed; every ability should be practiced in the most effective possible relations with other abilities [p. 140]." The reward that served to strengthen the practiced bonds was obtained when arithmetic problems were made interesting, fun, and close to practical applications. Thus, Thorndike was also concerned with the intrinsic meaningfulness of problems and their relevance to daily activities outside of school.

Some of Thorndike's bonds seem quite straightforward. It is easy for us to imagine that learning a bond like "2 + 2" (the stimulus) equals "4" (the response) would be enhanced by appropriate forms of drill. Arithmetic is full of these simple bonds. But not all arithmetic is so easy to translate into stimulus-response terms. As anyone knows who has tried to learn long division, some of it involves extremely long and complex operations. Thorndike (1922) explained these complex operations as "organized cooperating system(s) of bonds [p. 138]," groups of individual bonds that needed to be taught "teamwork" by being carefully sequenced and practiced, as described in the following quotation:

> As each new ability is acquired, then, we seek to have it take its place as an improvement of a thinking being, as a cooperative member of a total organization, as a soldier fighting together with others, as an element in an educated personality. Such an organization of bonds will not form itself any more than any one bond will create itself. If the elements of arithmetical ability are to act together as a total organized unified force, they must be made to act together in the course of learning. What we wish to have work together we must put together and give practice in teamwork [p. 139].

If, as Thorndike suggested, bonds were created by repeated pairing of stimuli and responses, then it seemed the teacher's job was merely to provide the proper amount of practice, in the proper order, on each class of problems. The teacher was to identify the bonds that made up the subject matter of interest and then put them in order, easier ones first, arranging them so that learning the easier ones would help in learning the harder ones that came later in the series. When that was accomplished, all that remained was to arrange for children to practice each of the kinds of bonds. Each class of bonds was to be practiced just enough so that errors could be avoided when advancing to the next harder class of bonds. The more often drill and practice could be presented in the context of interesting and practical problems, the stronger would be the connections. And since more complex problems were conceived as strings of bonds, it was important to drill well on each of the connections that would be needed for the harder problems.

Thorndike's books were rich with examples of the specific problems and drill sequences he recommended (an example appears in Fig. 2.2); but the rules for generating them were largely intuitive. Which bonds were easier? What was

32. A Percentage Race

Each row of pupils is a team. The teacher gives out printed problems, or uses those on these pages, or writes problems on the blackboard. All start together and write the missing numbers or answers as quickly as they can without making a mistake. At the end of 10 minutes all stop. The pupils interchange papers, mark with a cross each wrong result, and count the number of correct results. A pupil's score is the number of right answers with 2 off for each one wrong. The row with the highest average wins. Each pupil who makes any mistakes corrects them at home or during the study hour. Practice with this and the following page until you can make a good score.

1. 15% of \$1.50 = . . .	21. 1 1/2% of \$6000 = . . .
2. 12% of \$2.15 = . . .	22. 76 = . . . % of 380.
3. 20% of 80¢ = . . .	23. 22% of 25 mi. = . . .
4. 4% of \$300 = . . .	24. 4 = . . .% of 11.
5. 3 1/2% of \$16 = . . .	25. 1/2% of 600 = . . .
6. 1/2% of \$400 = . . .	26. 3% of 16 mi. = . . .
7. 105% of \$90 = . . .	27. 15% of 8 hr. = . . .
8. \$14 = . . . % of \$20.	28. \$25 = . . . % of \$130.
9. 39 = . . . % of 70.	29. \$32 1/3 = . . . % of \$40.
10. 56 = . . . % of 60.	30. 15 = 75% of . . .
11. 16 = . . . % of 25.	31. 2 1/2% of \$450 = . . .
12. 5 = . . . % of 7.	32. 3/4% of \$760 = . . .
13. 8 = . . . % of 9.	33. 45 = . . . % of 80.
14. 16 = 20% of . . .	34. 72 = . . . % of 80.
15. \$30 = 4% of \$. . .	35. 140 = . . .% of 215.
16. \$75 = 5% of \$ . . .	36. 122% of \$64.50 = . . .
17. \$5 = 10% of \$. . .	37. 18 = . . . % of 40.
18. \$12 = 6% of \$. . .	38. 1/8% of \$1000 = . . .
19. 6% of \$2000 = . . .	39. 21 = . . . % of 40.
20. 4 1/4% of \$24.50 = . . .	40. 21 = . . . % of 15.

FIG. 2.2 A sample drill lesson designed by Thorndike. Note the use of a "team race" approach, one way of strengthening bonds through reward. For this particular race, children were to complete 100 problems in 10 minutes. (From Thorndike, 1924.)

"enough" practice? What was the best way to organize practice on different kinds of bonds? These questions were not systematically addressed by Thorndike, but they generated a great deal of research on the psychology of arithmetic. As we see in a later section, this research continues to influence instructional practice even today. Thorndike thus took a big step in the direction of bringing psychological theory to bear on instruction. His contribution to the psychology of mathematics was to focus attention on the *content* of learning and to do so in the context of a specific subject matter. Thorndike's analysis of

arithmetic in terms of bonds to be strengthened in what children needed to learn verb ? in mathematics and how they could learn it.

DRILL VERSUS MEANINGFUL INSTRUCTION

Thorndike's theory of learning, and the drill method of instruction that it seemed to promote, was not without is detractors, even within associationist psychology. Although Thorndike stressed that drill problems should be made interesting and should be verified with concrete objects, the thrust of his influence was to sanction drill as the major method of arithmetic instruction. Early on, a strong voice was heard opposing the bond theory—that of William Brownell, among others. Brownell objected to the drill method, which he saw as a direct extension of the bond theory, on several grounds.

First, it took no account of qualitative differences in the computations of children and adults. If an adult could compute a grocery bill by directly recalling a few addition facts from memory, did this mean a child should be taught to do it exactly the same way? Brownell thought not. When he interviewed third graders in drill programs (Brownell & Chazal, 1935/1958), he found them using a variety of procedures other than direct recall to do their addition and subtraction problems. They counted on their fingers; they solved from known combinations (e.g., I know that 4 + 4 is 8, so 4 + 5 is 9); or they gave immediate answers, but incorrect ones, which indicated they were simply guessing. The children were using these methods even after having been drilled for 2 years in the number combinations. Brownell interpreted this to mean that drill simply made them faster and better at the "immature" procedures they had discovered for themselves, not at the kind of direct recall that adults possess.

Second, the drill method implied a distorted view of the goal of learning. For Brownell and others of his persuasion (e.g., Wheeler, 1939) the criterion of arithmetic skill was the ability to think quantitatively, not to respond with 100% accuracy to a given list of arithmetic problems. Said Brownell (1928):

> The child who can promptly give the answer 12 to 7 + 5 has by no means demonstrated that he knows the combination. He does not "know" the combination until he understands something of the reason why 7 and 5 is 12; until he can demonstrate to himself and to others that 7 and 5 is 12; until he is so thoroughly convinced that 12 is the right answer for 7 + 5 that he can give it as the answer with assurance of its correctness; and until he can use the combination in an intelligent manner—in a word, until the combination possesses meaning for him [p. 198].

To ensure this sort of meaningfulness through instruction required some attention to the mathematical principles and patterns underlying computations. "If one is to be successful in quantitative thinking," Brownell said, "one needs a fund of meanings, not a myriad of 'automatic responses.'... Drill does not

develop meanings. Repetition does not lead to understandings [Brownell, 1935, p. 10].'' For example, asking children to recite ''2 + 2 = 4'' over and over again did not guarantee they could understand an important underlying number concept: The symbol ''2'' refers to the attribute of ''twoness'' that characterizes a set comprising that many objects; ''2 + 2'' represents the operation of combining two sets to create a set having the attribute of ''fourness.'' Without plenty of experience combining and taking apart sets of concrete objects, a child reciting number combinations perfectly was just making ''correct noises,'' Brownell thought. Instruction that stressed concepts and relationships was seen as the way to ensure skilled quantitative thinking. Though perhaps more roundabout than drill methods in achieving speed and accuracy, the meaningful method would achieve something more important. It would ''help pupils organize and unify their knowledge of number, to develop facility in dealing with numbers, and to understand the principles of number combination [Brownell, 1928, p. 211].'' Without meaningful instruction to point out the interrelationships, drill would encourage students to view mathematics as a ''mass of unrelated items and independent facts.''

Given the proper understanding of mathematical concepts and procedures, Brownell went on to say, students would be better able to apply their knowledge in novel situations. There was research to back up this assertion. For example, McConnell (1934/1958) compared a rigorous drill method (sheer repetition of abstract symbols) to a meaningful method of instruction. In the latter, the number facts were presented in conjunction with pictures or objects and then practiced with opportunities to verify answers. He found drill to be ''the more forthright means of attaining automatic and immediate responses to the number facts,'' but on measures of transfer to untaught combinations, the meaningful approach gave significantly better results.

Later, Swenson (1949/1958) compared three methods for teaching the addition combinations to second graders. The *drill* method featured interesting, varied drill exercises; avoided excessive rehearsal of errors; and provided for extra practice on difficult combinations. The *generalization* method encouraged children to apply their learning to new problems at every phase and allowed them to continue counting and solving methods until they could switch comfortably to direct recall. The *drill-plus* method combined drill with some meaningful instruction: Number combinations were verified using counting and concrete manipulations; facts were arranged into groups that yielded the same answer, so as to highlight mathematical patterns. The generalization method proved most effective in promoting both learning of the material and transfer to new material, with drill-plus somewhat less effective, and straight drill least effective.

Presumably, at some point even children taught by ''meaningful'' methods had to practice number combinations and computational procedures so they could recall them accurately and quickly. Although Brownell declined to elaborate on the form, content, and timing of practice, focusing instead on the need for reform in arithmetic instruction, there did seem to be a place in his instructional scheme for some sort of practice. In concluding an extensive study of children's number concepts, Brownell (1928) hypothesized that computational proficiency developed in three overlapping phases: First, the child learned a procedure for executing a calculation—any procedure, be it counting on one's fingers, solving from known facts, or direct recall. There followed a period of increasing accuracy in the execution of the procedure, later accompanied by a rapid increase in the speed of calculation. Any switch to a new procedure—say, from addition by counting to addition by solving from known combinations—would be attended by an initial decline in accuracy and speed until the new procedure became familiar. This suggested one might introduce drill for accuracy and speed, but only following a period of familiarization with the computational process. Even so, this drill would have to be ''meaningful habituation'' rather than simple repetition. Practice, according to Brownell, would only be worthwhile if it included exercises to increase understanding.

Where Brownell and Thorndike differed was in their definitions of what should be learned. To Thorndike, mathematical learning consisted of a collection of bonds; to Brownell, it was an integrated set of principles and patterns. The two definitions in turn seemed to call for very different methods of teaching, either drill or meaningful instruction. Today most educators acknowledge the need for both types of learning experiences, but how they should be integrated is still not clear. In the meantime, a number of psychologists have elaborated the notion of meaningfulness in learning. In Chapters 5 through 7 we present more systematic theoretical justifications for meaningful instruction than those Brownell was able to offer, and we give some examples of what such instruction might entail. There we also consider a further argument against strict drill, namely, that it habituates children to an unthinking, inflexible mode of response, whereas mathematical thinking often demands flexibility and creativity.

WHAT MAKES ARITHMETIC PROBLEMS EASY OR HARD?

While drill methods came under increasing attack from psychologists of Brownell's persuasion, other psychologists applied their energies to the pursuit of better, more effective forms of drill. Thorndike and others examined the textbooks of the day and found them varying widely in the amount of practice given on the different number combinations. This spurred a period of research on the relative difficulty of arithmetic problems. The object was to be able to provide the proper amount of practice: less practice on easier problems, more practice on harder ones.

Knight and Behrens (1928), for example, monitored the behavior of 40 second-grade students as they learned and practiced the 100 addition and 100

subtraction combinations. These were the number combinations having a sum of less than 20, such as 1 + 2, 6 + 5, or 19 − 7. Knight and Behrens kept extensive records on the numbers of errors made, the numbers of exposures needed to master each combination, the amount of time needed to solve each combination on each occasion, and the amount of practice necessary to maintain proficiency. They considered children to have mastered the combinations as soon as they could reliably do them all with close to 100% accuracy, with no apparent hesitation, and with the same accuracy and speed after 3 weeks' elapsed time. The product of this study was a complete list of all the addition and subtraction combinations arranged in their order of difficulty—translatable into the amount of practice a teacher should require on each one. In subtraction, for example, they found that the hardest combination, 15 − 6, took children an average of 26 trials to master, at about 7.4 seconds per problem, with an average of 20 errors. A sample of their addition ranking is shown in Fig. 2.3. An interesting sidelight is that one could not assume that the reversed forms of number combinations were of comparable difficulty; in fact 9 + 5 ranked ninetieth, whereas 5 + 9 was the hardest of all; and 3 + 0 ranked twentieth, whereas 0 + 3 was the easiest of all.

To Brownell, data like these furnished grounds for criticizing the bond theory. If every single bond had to be individually taught—not only 5 + 9 but also 9 + 5 and later 9 + 15 and 9 + 25—then the task of instruction was absurdly immense. In the context of Knight and Behrens' drill program, children apparently did not learn, or did not know how to apply, the mathematical principle of commutativity. Later, we present evidence that, with the proper instruction, even preschoolers can demonstrate an understanding of commutativity.

Other researchers (e.g., Clapp, 1924; Wheeler, 1939) developed their own ordered lists of number combinations, and there were arguments over which list was more valid. The results of such studies were used to organize drill and practice in schools. Easier addition and subtraction combinations were introduced earlier—without necessarily giving attention to the logical relationships among combinations—and textbooks and exercise pages devoted greater or lesser amounts of space to each combination according to its ranking on someone's list. The same was true for the multiplication combinations (Norem & Knight, 1930).

Studies that simply ranked problems did not tell much about *why* the problems were easier or harder. An analysis of the patterns of ease and difficulty points to some possible explanations, however. For one thing, the combinations involving sums over 10 clustered around the harder end of virtually every list of addition problems. For another, subtraction problems took longer to solve and involved more errors than addition problems. Wheeler (1939) was able to show that problems having a common addend (e.g., all problems in which 7 is added to another number) were of approximately equal difficulty and that difficulty increased with the size of the addend. These findings suggest differences in the

Addition Combinations	Difficulty Rankings: Knight-Behrens (1928)	Clapp (1924)	Wheeler (1939)
5 + 9	100	90	82
7 + 9	99	99	100
8 + 7	98	92	97.5
5 + 8	97	98	97.5
8 + 9	96	89	99
9 + 7	95	97	94
7 + 8	94	93	91
8 + 5	93	100	84
4 + 9	92	85	78
6 + 8	91	96	96
.	.	.	.
.	.	.	.
.	.	.	.
9 + 2	55	9	64
3 + 8	54	70	66
2 + 5	53	23	43
2 + 7	52	63	45
8 + 2	51	34	48
5 + 4	50	12	53
2 + 9	49	24	59.5
2 + 4	48	40	50.5
4 + 0	47	18	22
7 + 7	46	15	46.5
.	.	.	.
.	.	.	.
.	.	.	.
2 + 2	10	3	37.5
6 + 0	9	36	13
0 + 7	8	50	17
0 + 4	7	43	13
0 + 1	6	49	2.5
2 + 0	5	14	9
9 + 0	4	75	6
1 + 0	3	59	9
1 + 1	2	10	9
0 + 3	1	54	32

FIG. 2.3 A sample of the difficulty rankings on the 100 addition combinations. Although used to specify the proper amount of practice on each combination, difficulty rankings also indicated possible differences in mental solution processes.

amount of mental processing required to do larger sums as compared to smaller or to do subtraction problems as compared to addition. The fact that a sum like 12 + 7 consistently takes longer or yields more errors than 7 + 2 suggests children may be using different procedures to find the two answers. Brownell's interviews with children revealed some of these possible procedures, and today psychologists are analyzing children's solution procedures at even finer levels of detail. In

Chapter 4 we examine several different counting procedures that children use when they cannot recall the combinations directly from memory. The various procedures take different amounts of time depending on how large the number(s) to be counted are. These differences in processing may account for patterns of difficulty such as Wheeler (1939) found.

Alongside studies ranking the various arithmetic combinations, there was another line of research that attempted to rank problems on the basis of various measures of problem complexity. This research involved story or word problems, which are often used to give practice in specific computations, in general problem-solving skills, or in applications to real-life tasks. A word problem is a computation requested in verbal form, for example:

> Bob buys two boxes of nails at $1.00 each and 3 gallons of paint at $4.00 a gallon. If he had $20 to start with, how much does he have left to spend on Mary when he takes her to the movies this Saturday night?

Such a problem exercises multiplication (2×1, 3×4), addition ($2 + 12$), and subtraction ($20 - 14$) skills. It also gives practice in translating stories about real-life situations into mathematical problems.

The difficulty of word problems was shown to be affected by many factors. In a number of studies (Brownell & Stretch, 1931; Hydle & Clapp, 1927; Kramer, 1933), researchers presented children with word problems requiring identical operations on identical numbers but varying in wording and specific problem contexts. Some of the variables that seemed to contribute to problem difficulty, as determined by the number of errors children made, were the familiarity of the situations described in the problems; the arrangement of problems in a series; the number of unfamiliar objects and nonessential elements; whether the story problems were intrinsically interesting; how difficult the vocabulary was; whether the problems were presented as declarative statements or as questions; and so forth. On the basis of their research, Hydle and Clapp (1927) argued that word problems should involve familiar situations that children could easily visualize (e.g., a circus as opposed to an African diamond mine) as a first step in problem solving. Brownell and Stretch (1931) countered with evidence that situations did not need to be familiar, provided the other complicating variables were kept at a minimum level. In fact, they argued, some exposure to unfamiliar situations was desirable so that children would learn to appreciate the wide applicability of number operations. Although these studies were inconclusive, they did point to factors that teachers should be aware of in preparing or assigning word problems for student practice.

More recent studies have continued and systematized this line of effort, taking advantage of the data gathering capacities of computer-assisted instructional programs. Loftus and Suppes (1972), for example, predicted problem difficulty on the basis of what they called "structural variables," or characteristics of individual problems that seemed to contribute to their complexity. Each structural variable was quantified, and problems were assigned a difficulty rating accordingly. Sixth-grade students then worked on 100 word problems at a computer terminal that recorded all their solution attempts, correct and incorrect. In subsequent statistical analyses, certain of these variables were found to be particularly influential in determining story problem difficulty, as measured by the probability of a student's getting the correct answer: (1) the number of different arithmetic *operations* needed to arrive at the solution; (2) the *sequence* variable, or whether the problem was solvable by the same operations, in the same order, as the previous problem; (3) the *length* of the problem, or number of words in the problem statement; (4) the *depth* variable, or the grammatical complexity of the wording; and (5) whether or not a *conversion* of units of measurement was required. The effects of other hypothesized structural variables—required number of steps to solution, presence of verbal cues such as "and," "left," and "each" (signaling addition, subtraction, and multiplication or division, respectively), and the order in which information was presented within a problem—were not significant.

By the criterion of structural variables, the sample word problem about Bob and Mary should be judged fairly difficult. It requires three different operations (multiplication, addition, and subtraction), is 45 words long, contains embedded phrases as well as irrelevant information (Mary has nothing to do with the computational aspect of Bob's problem). However, it might be perfectly appropriate as a practice problem for a fifth or sixth grader. Contrast our statement of the problem with the way it might appear in some of the "pared-down" word problems in recent primary textbooks:

> Bob buys: 4 boxes of nails, $1 each
> 3 gallons of paint, $4 each
> Bob had $20. How much is left?

The same number of operations is required, but the number of words is considerably reduced and the grammar less complicated. The problem should be easier to solve and, therefore, more appropriate for younger or less skilled students.

The point of identifying these structural variables was to be able, eventually, "to formulate a clear set of rules or a formula for generating sets of arithmetic problems of a specified difficulty level. Curriculum developers would then be in a better position to control difficulty level when preparing instructional materials [Jerman & Rees, 1972]." In other words, the object was to design better practice sequences. At the same time, this research provided clues to the mental processing involved in certain kinds of problem solving. Knowing which variables affected problem difficulty suggested the kind as well as the number of mental steps needed for problem solution. But no theories specifying the steps involved in processing word problems were actually developed in the course of this research. In later chapters, we consider how detailed study of the steps by which story problems and other computations are solved has helped build psychological

theories to explain human thought processes in general and mathematical thinking in particular.

OPTIMIZING THE EFFECTIVENESS OF PRACTICE

The relative difficulty of arithmetic problems is only one factor, albeit an important one, in organizing drill and practice. In addition to deciding which problems should be practiced first and which ones later, there are decisions to be made about when to introduce and terminate practice, when to switch to a new level of difficulty, what kind of practice to give, and the like. These questions accept the assumption that practice is important to becoming proficient in computation and they focus on organizing computational practice to optimize its effectiveness.

Psychologists have long been interested in optimizing practice on all kinds of skills, not just arithmetic. It is generally established, for example, that *spaced* practice is more effective than *massed* practice for most skills; that is, practice sessions of moderate duration spaced out over several days produce better learning than the same amount of practice concentrated into one long session. The superiority of spaced over massed practice for arithmetic in particular has been demonstrated experimentally by Buswell (1930) and Repp (1930, 1935). Repp was concerned with the form drill should take if one wanted to maintain children's proficiency in basic arithmetic operations on whole numbers and fractions. Should students be given extended practice involving only similar types of problems (''isolated'' drill) or should different types of problems be interspersed with one another (''mixed drill'')? Most textbooks of the day employed only isolated drill, but Repp felt there was little experimental evidence to support that method. He designed a study to explore the relative merits of isolated and mixed drill.

Repp tested 538 12-year-olds and created two groups matched on three levels of ability (low–average–high). One group of children practiced on blocks of similar problems, each week concentrating on one particular computational skill. This was the isolated-drill group. The mixed-drill group, on the other hand, practiced a variety of computational skills each week but worked only one or two problems of each type. Examples of the two forms of drill are shown in Fig. 2.4. Practice sessions took place once a week for both groups and lasted 20 minutes. After 26 weeks, both groups had received the same amount of practice on each type of problem, the only difference being the organization of practice. However, although both groups had gained in speed and accuracy, the mixed-drill group showed a 23% greater gain than the isolated-drill group. Mixed drill yielded greater speed and accuracy at all ability levels—low, average, and high alike—with the greatest gain (54%) appearing in the low-achievement group.

This experiment seemed to indicate that in general it was better to practice a skill in small amounts and frequently than to practice in great amounts less often.

The Isolated Type of Drill Organization

1. 2/9 + 5/9 =	2. 1/2 + 1/2 =	3. 5/7 + 2/9 =	4. 2/3 + 3/8 =
5. 9/10 + 1/5 =	6. 5/9 + 3/7 =	7. 8/9 + 1/6 =	8. 4/5 + 1/2 =
9. 2/3 + 5/6 =	10. 3/5 + 7/8 =	11. 3/5 + 2/7 =	12. 8/9 + 1/9 =
13. 7/8 + 2/3 =	14. 3/4 + 5/6 =	15. 3/5 + 1/2 =	16. 8/9 + 7/8 =
17. 1/2 + 1/4 + 1/3 =	18. 3/4 + 5/8 + 1/2 =	19. 5/7 + 2/9 + 3/5 =	20. 1/8 + 1/7 + 1/4 =

The Mixed Type of Drill Organization

1. 14) 8599	2. 6 1/2 − 1/3	3. 27) 538	4. 8963 x 38
5. 1/3 x 12 x 5 =	6. Subtract: 9 1/2 from 10 4/5 =	7. 80856 − 77184	8. 976965 9766 85 27234378
9. Multiply: 438 by 577	10. 295 x 389	11. 4 2/3 + 7 1/2 =	12. 3/10 ÷ 2/9 =
13. 84 31 99 46 77 665 3477 89 68	14. 3/4 x 2 x 1/3 = 17. 9912 − 7383	15. 65) 1823	16. Add: 998 239 234 910 629
	18. Subtract 4 bu. 2 pk. 1 qt. from 8 bushels. Answer................		
	19. Add: 94 1/2, 2/3, 26, 10 3/4		20. Multiply: 4209 63

FIG. 2.4 Examples of isolated and mixed drills. (From Repp, 1935.)

Nevertheless, after further study, Repp (1935) acknowledged that isolated drill seemed to play an important role in firmly establishing new learning. He suggested using isolated drills ''in close proximity to the first teaching and learning of new facts, skills, and information,'' reserving mixed drill for maintenance purposes once the new skill was well learned. Isolated drill also seemed useful as an antidote to habitual wrong responses on particular computations, for

example, in the case of a student who habitually reported 75 as the product of 9 × 8. Despite the many studies that existed on drill and practice at the time, including his own, Repp (1935) was able to conclude only that, "Generally speaking, the proper use of drill in arithmetic needs to be better understood. Correct drill construction and its use is still a fertile field for research [p. 200]."

Computer-Assisted Drill. The search for better organized, more effective drill programs continues into the present day. A new thrust has been added, however. With the availability of computer technology, some researchers have explored the possibility of tailoring drill programs quite precisely to the ability levels of individual children, and in so doing to optimize conditions for improving computational skill. Because of the computer's capability to interact directly with the learner and to store detailed data about the learner's performance, it has quickly been adopted as an experimental medium for individualizing drill and practice as well as other types of instruction. The resulting computer-assisted instructional (CAI) programs provide a unique opportunity to study the mathematical performances of children. As we see in later chapters, the computer has been an important tool in a variety of studies on human learning.

One series of drill-based programs that combined instruction with research on optimal approaches to teaching was designed by psychologists at Stanford University's Institute for Mathematical Studies in the Social Sciences. The Stanford arithmetic programs resulted from a project that used the computer as the basis for providing individual practice on the computational skills customarily taught in grades 1-6. (For a detailed description of these programs, see Suppes, Jerman, & Brian, 1968; Suppes & Morningstar, 1972.) Like the earlier noncomputerized experiments that studied mathematics learning in the context of instruction (e.g., Wheeler, 1939; Knight & Behrens, 1928), the Stanford work assumed that practice was essential if children were to become fluent and competent performers of computational tasks. And like the earlier studies, the Stanford programs assumed that drill should proceed roughly from easier to harder kinds of problems, include an element of speed demand, and strive for a high and lasting degree of accuracy.

There was nothing unusual about the content of the Stanford CAI programs. Like most standard school mathematics curricula, they contained practice in addition, subtraction, fractions, multiplication, long division, percents and ratios, and so on at various levels of difficulty appropriate to different age groups. However, unlike conventional programs, the computer was able to give continuing feedback, so that children could tell immediately whether their answers were right or wrong. It also had a built-in decision-making apparatus that allowed for adjustments in problem presentation and difficulty level. This meant children never had to work on problems that were too easy or too hard for them. Thus the CAI program represented a conscious attempt to enhance children's motivation while avoiding practice in error.

Briefly, this is how the programs worked from the point of view of the child and the teacher. In or near each participating classroom, a computer terminal was made available to students. Each child was scheduled to spend 5 to 15 minutes at the terminal once a day. Drill was organized into "concept blocks," to be completed in 3 to 12 days, that supplemented similar content taught in a textbook series. Each concept block contained problems of roughly the same type (e.g., sums from 0 to 20, multiplication tables 2 and 3, or units of measure), although many were "mixed reviews." Each included a pretest, 5 days of drill, a posttest, and sets of review drills and review posttests. The teacher assigned concept blocks appropriate to the material the class was covering in its arithmetic textbook.

Since all children in a class worked on the same concept block, they were all practicing roughly the same arithmetic content. But within each concept block the computer was able to present problems of five levels of difficulty; based on an individual child's performance, it could make a decision about which level of difficulty would be appropriate for practice. Each child started out at a level determined by his or her pretest score. On succeeding days, the difficulty of problems was adjusted according to the child's practice score from the day before. Children scoring between 60 and 79% stayed at the same level the next day; children scoring 80% or above moved to the next hardest level; those scoring below 60% were given easier problems. Review drills were "chosen" by the computer program so that individual children had extra practice in content on which they had scored least well in the past.

Feedback was given on each problem, as shown in the sample drill in Fig. 2.5. If a child's answer was correct, the computer printed out a message that said so; if a child gave the wrong answer, the computer reported the error, let the child try again, and after a certain number of tries printed out the correct answer. Further, if a child took too long to solve a particular problem, the computer gave a "time is up" signal to encourage the child to work at a reasonably fast pace.

We suspect that Thorndike would have been pleased by the possibilities for well-organized drill that the computer-assisted programs represented. Precisely tailored drills minimized the problem of "stamping in" incorrect bonds by forestalling errors as much as possible. Bonds that were more difficult—as determined by error data from daily performance—received the additional practice they required. But instead of determining the difficulty of bonds according to a subjective standard or some class average, the computer interpreted the difficulty of bonds according to the individual practicing them. Implicitly, it acknowledged that certain bonds might be harder for some people than for others. It was also able to respond to individual differences in previous knowledge and general aptitude.

Did the Stanford drill programs work? Data gathered on the Stanford programs during the most intensive period of study, 1965 to 1968, were used to

```
PLEASE TYPE YOUR NAME
MIKE ODELL
          DRILL NUMBER 509013
     (42 + 63) / 7 = (42 / 7) + (63 / 7)
     48 - 38 = 38 - 48
WRONG
     48 - 38 = 38 - 4
WRONG, ANSWER IS 28
     48 - 38 = 38 - 28
     76 - (26 - 10) = (76 - 26) + 10
     4 X (7 + 13) = (4 X 7 ) + (4 X 13)
     (53 - 20) - 11 = 53 - (20 + 11)
     32 + (74 + 18) = ( 32 + 74) + 18
     51 X (36 X 12) = ( 51 X 36) X 12
     17 X (14 + 34) = (17 X 14) + (17 X 34)
     362 + 943 = 943 + 362
     (5 + 8) X 7 = ( 5 X 7) + ( 8 X 7)
     (90 / 10) / 3 = 90 / (10 X 3)
     (72 / 9) / 4 = 72 / ( 9 X 4)
     (54 + 18) / 6 = (54 / 6) + (18 /__)
TIME IS UP
     (54 + 18) / 6 = (54 / 6) + (18 / 6)
     60 - (19 - 12) = (60 - 19) + 12
     72 X (43 X 11) = (72 X 43) X 11
     (63 / 7) + (56 / 7) = ( 63 + 7 ) / 7
WRONG
     (63 / 7) + (56 / 7) = ( 63 + 56) / 7
          END OF DRILL NUMBER 509013
     13  MAY  1966
     16 PROBLEMS
                         NUMBER     PERCENT
     CORRECT               13          81
     WRONG                  2          12
     TIMEOUTS               1           6
     WRONG
     2
     16
     TIMEOUTS
     13
          222.7 SECONDS THIS DRILL
     CORRECT THIS CONCEPT - 81 PERCENT, CORRECT TO DATE - 59 PERCENT
     4 HOURS,  46 MINUTES,  59 SECONDS OVERALL
     GOODBYE MIKE.
```

FIG. 2.5 Sample printout from the Stanford computer-assisted drill programs. This represents one drill session completed by a fifth grader. (From Suppes, Jerman, & Brian, 1968. Copyright 1968 by Academic Press. Reprinted by permission.)

analyze the performance of almost 4000 children, grades 1-6, in selected sites in four different states. The Stanford Achievement Test (SAT) scores of children who had participated in the CAI program were compared with the scores of control groups who had simply received standard classroom instruction. On most of the sections of the SAT test, and in most grades, the CAI children improved significantly more than did the control children. This was true even for tests that stressed concepts and applications rather than strictly computational skills. And this effect was obtained with only 8 months of work, and only 5 to 8 minutes per day at the computer. But these results were obtained using experimental groups who received classroom instruction combined with supplementary drill on the computer, whereas the control groups had only regular classroom instruction and no extra drill. What if the control classes had received supplementary drill in the form of paper-and-pencil practice exercises?

Through an unforeseen occurrence, the Stanford study was able to suggest an answer to this question as well. One of the control schools found their pretest scores were so poor that they initiated their own paper-and-pencil drill sessions. Children who had this extra work for 8 months actually did better on the posttest than the CAI children. This finding agrees with other more recent, deliberately planned evaluations of CAI versus paper-and-pencil drill programs (Jacobson, 1975). However, these control children had to put in 25 minutes per day of extra work, compared to the 5-8 minute computer sessions, and the teacher had to spend extra time grading their papers. Nevertheless, these results do suggest that the practice itself, rather than the fact that it was occurring on a computer terminal, produced the improvement in performance.

The Stanford data show that well-planned drill and practice can increase accuracy in computations of many kinds for children of many ages. However, children who started out with better accuracy scores improved about as much as did the children who started out lower in accuracy. Thus, although the programs adapted to individual differences, they did *not* have the hoped-for effect of helping the initially less competent children "catch up" with the more competent ones. Instead, all children improved in most areas. What is more, some concepts were very hard to improve within the confines of the practice programs. This suggests that drill will not improve all aspects of arithmetic performance, not even drill that responds to individual differences.

It is noteworthy, too, that the Stanford programs rarely demonstrated accuracy greater than 90% within concept blocks (Suppes & Morningstar, 1972), a result consistent with other learning experiments conducted in the laboratory (Judd & Glaser, 1969). In a study on computer instruction in simple number combinations, Jacobson (1975) demonstrated that once accuracy reached a certain level, 90-95%, further practice could bring about no significant increase. This is quite far from the 995 to 997 out of 1000 mastery Thorndike thought could be achieved! These studies suggest it may never be possible to achieve perfect

accuracy, at least for children. Teachers and instructional designers thus need to be sensitive to the point beyond which additional drill has no particular value.

DRILL AND THE DEVELOPMENT OF AUTOMATICITY

Much of the theory and research presented so far has concerned the role of drill in promoting speed and accuracy. We can assume that accuracy is important in computation, but one might well ask whether speed is a particularly important goal of instruction. As we have noted and as we develop further in the second half of this book, some psychologists and educators argue that instruction on even the simplest, most basic arithmetic skills should help children understand mathematical concepts rather than simply memorize facts and procedures. If understanding is established, they argue, children can reconstruct the forgotten items or even construct their own procedures for finding answers when memory fails. Where the criterion of mastery is grasp of ideas, speed of recall seems comparatively unimportant.

The counterargument holds that, on the contrary, it is very important for children to memorize certain facts and procedures to the point where they need not think about them but can do them rapidly and almost automatically in the course of computation. The function of drill, according to this view, is to develop automatic responding, indicated by a high rate of speed. Automatic responding has been talked about in psychology since before the days of Thorndike (e.g., Huey, 1908, on the role of automatic responses in reading). However, recent developments in psychological theory now make it possible to be more specific about what we mean by automatic responding—or *automaticity,* as it is often called.

A convenient framework for considering automaticity, and one that we call upon in the ensuing chapters, is the information-processing view of the human mind. In this view all human behavior is seen as the result of the mind acting upon (processing) data from the internal or external environment (information). Although there are many differences in detail, virtually all psychologists working within the information-processing framework hold a common assumption about the structure or "architecture" of the human mind. The assumption is that information is processed through a series of "memories," each capable of different kinds of storage and processing and each subject to different limitations. Together these memories constitute the information-processing "system."

Working from the outside in, information first enters the system through a *sensory intake register* (sometimes called a sensory buffer or iconic memory). This first memory in the system can receive visual, auditory, and tactile information directly from the environment, and it can take in a lot of information simultaneously. But it can hold that information for only a very short time—less than 1 second. If in that time the other components of the memory system fail to

attend to the information in the sensory intake register, that information is lost. The component that does this "attending" is called *working memory* (or sometimes short-term memory or intermediate-term memory). This is where the actual thinking gets done, that is, where operations are performed on information. The third component of the system is *long-term memory,* where everything a person knows is stored.[1]

Within this general structure, working memory plays a crucial role. Only by being processed in working memory can information from the sensory part of the system enter a person's long-term memory store. And only when information is called out of the long-term store into working memory can the stored information be used in the course of thinking. Like the sensory intake register, working memory has a limited processing capability. Working memory is limited not by the length of time in which it can retain information but by the amount of information it can handle at any one moment. No one knows exactly how many "pieces" of information it takes to "fill up" working memory, although about seven pieces, give or take two, has been long proposed as the capacity of adults' working memories (Miller, 1956). If working memory is "full," then new information coming in from either the sensory system or long-term memory is accepted, but older information is lost. In general, the information that has been least recently attended to is the information that will be lost. Rehearsal (repeating the information to oneself from time to time) can make it possible to retain information in short-term memory. But rehearsal cannot increase the basic capacity; it cannot extend the number of memory "slots" there are to be filled. One way to extend working memory's processing capacity, however, is to organize small pieces of information into "chunks," so that each slot is in fact filled with more information. The larger the chunks, the more information working memory can handle. We explore these organizing or chunking processes later in Chapters 4 and 8.

Another way to extend the capacity of working memory is by developing automaticity of responding. The argument is as follows: To the extent that certain processes can be carried out automatically, without need for direct attention, more space becomes available in working memory for processes that *do* require attention. To relate automaticity to the domain of computation, we need to distinguish between two kinds of arithmetic tasks on which drill and practice is commonly given. On the one hand we have the so-called *number facts,* that is, the single-digit number combinations that form the basic building blocks of all computations. Number facts are of four kinds—addition, subtraction, multiplication, and division (e.g., $5 + 4 = 9$, $8 - 3 = 5$, $9 \times 8 = 72$, $8 \div 2 = 10$). On the other hand, we have *algorithms,* or procedures of computation. These are the

[1]An alternative conception (Craik & Lockhart, 1972) plays down the separate character of these memory stores in favor of the notion of successively deeper "levels of processing," but all the fundamental memory operations must still be carried out.

sequences of operations that we perform, using the number facts, to arrive at solutions to more complex problems. For example, to add columns of digits that sum to more than 10, we perform a series of smaller operations:

To add:	20	We do the following:
	18	Add the ones column ($0 + 8 = 8$; $8 + 4 = 12$; $12 + 3 = 15$).
	4	Notate the 5; carry the 1.
	3	Add the tens column (1 [carried] + $2 = 3$; $3 + 1 = 4$).
		Notate the 4; answer 45.

This procedure is an algorithm. To take another example, long division involves estimating divisibility, multiplying, subtracting, bringing down the next number, dividing again, and so forth. This is the long-division algorithm.

Suppose Susan must execute an algorithm involving several steps, such as adding fractions of different denominators:

$$3\tfrac{1}{2} + 8\tfrac{1}{4} = ?$$

The actual sequence of steps is long, and an information-processing psychologist would say it requires holding a great deal of information in working memory, particularly if it must be done without paper and pencil:

> $(3 \times 2) + 1 = 7$, over 2, change to fourths, equals 2×7, over 4, or 14 fourths; plus $(8 \times 4) + 1$, over 4, or 33 fourths; $14 + 33$ over 4, etc.

While Susan carries out this calculation, much of working memory will tend to be occupied with keeping her place in the long procedure and holding the numbers needed for several intermediate computational steps. It is easy to see in this case why it might be desirable for Susan to have an automatic command of the number facts. If space must be taken up in working memory by the computation of 8×4 or 2×7 or by an extended search for these facts in long term-memory, then place-keeping functions and memory for intermediate computations are likely to meet serious interference. What is more, procedural errors are likely to creep in unnoticed. On the other hand, if practice has rendered those number facts instantaneously retrievable from long-term memory, then working memory is permitted to function more efficiently.

The same holds for automatic access to memorized procedures or algorithms. If Susan has to reconstruct how to change fractions into their lowest common denominator each time this procedure is needed, then valuable space is again taken up in working memory by processes that might well have become automated through proper practice. Thus emerges the strong suggestion that at least certain basic computational skills—number facts and simple algorithms—need to

be developed to the point of automaticity so they can avoid competing with higher-level problem-solving processes for limited space in working memory.

Direct evidence regarding the effect of automaticity on arithmetic calculation is not yet available. However, there exists analogous evidence in the psychological research on reading. It seems that automaticity of word recognition skill is associated with higher levels of reading comprehension (LaBerge & Samuels, 1974; Perfetti & Hogaboam, 1975). Perfetti and Lesgold (1979) encountered three types of word recognition skill in young readers, each of which suggested a different instructional treatment: (1) slow and inaccurate; (2) slow but accurate; and (3) fast and accurate. If we extrapolate to the domain of mathematics, we might find similar levels of skill among, say, learners of number facts. We could predict what kind of practice would benefit each type, as follows: Students demonstrating slow, inaccurate mastery of number facts would need practice that stressed accuracy and perhaps some help in understanding the reasons for particular procedures. Fast, accurate students, on the other hand, would need no isolated practice at all. The slow but accurate students would be candidates for exercises specifically designed to develop automaticity, for example, speeded practice. Of course, we are speaking hypothetically, based on studies in the domain of reading. But there is promise in this approach for new insights into the enhancement of mathematical skills.

It is clear from recent surveys that many people, even adults, perform very poorly in arithmetic calculation. Teachers seem to feel this is due to incomplete or inadequate mastery of number facts (Jacobson, 1976; Lankford, 1972), and psychological studies support this contention (Anaspaugh, cited in Buswell, 1927; Tait, Hartley, & Anderson, 1973). In other words, when children's algorithmic computations are examined, the correct processes have often been carried out in the proper order, but specific errors in recall of number facts produce errors in the final answers. However, in at least one study (Jacobson, 1975) that compared different types of drill, including CAI, accuracy of performance on number facts did not seem to predict performance on more complex algorithms. This suggested that errors in algorithmic computations could not be attributed solely to not knowing number facts. Jacobson sought to explain the discrepancy by the fact that isolated practice of simple number combinations presented unique learning and memory problems that do not exist when the combinations are used in actual problem-solving situations. In our view, another contributing factor could have been a lack of automaticity for number facts. Slow, accurate students, for example, might easily have passed accuracy tests on number facts but would still have failed in complex computations because their command of number facts was not automatic enough to avoid placing a heavy processing load on working memory.

A comment is in order concerning the way some of the recent innovative mathematics curricula incorporate drill and practice. The ideal espoused by the curriculum reform movement of the early 1960s (*Goals for School Mathematics*,

1963) was to avoid drill as much as possible by building it into the curriculum in a "spiral" fashion. Instruction in a new procedure or skill would be followed by a limited amount of drill on that skill by itself. Additional experience with that skill would later be provided, at various points in the curriculum, in the context of other types of problems of increasing complexity. The hope was to avoid the pitfalls of drill but to ensure enough practice so that competence in arithmetic skills was achieved.

The research we have presented suggests dangers that teachers should be aware of in the spiral approach. Students who fail to commit specific number facts or algorithms to memory the first few times they are presented may never "catch up" if the only further practice they receive is embedded in more complex problems. Imagine children trying to keep their minds on the complicated steps in finding square roots if they had to stop repeatedly and think what 6×7 was or how they were supposed to do "borrowing"! Drill and practice as a supplement to instruction—for the specific purpose of developing automaticity—may be of immense help from time to time for certain students, just to forestall such a pattern of failure and frustration.

One would like to be able to state a definitive set of rules for administering drill and practice. Unfortunately this would be premature. There are indications that drill does increase the speed and accuracy of responses, and we have pointed to one reason why speed might be particularly important in certain types of calculation skills. However, the research surveyed to this point leaves many questions unanswered. There are questions concerning what children learn: Do they learn bonds or patterns? Do these seemingly different types of knowledge demand different forms of practice? Could practice on a variety of related bonds eventually contribute to understanding larger patterns? There are questions concerning when practice should be introduced: Should it come before, during, or after more "meaningful" instruction in the conceptual underpinnings of computational procedures? Could repetition actually contribute to conceptual understanding? There are serious questions concerning what drill really accomplishes: Does drill teach children to do immature procedures faster, or does it push them to learn more efficient procedures? And how does the switch from one procedure to another come about? These issues lead us head on into the question that concerns us throughout this book: What goes on in children's heads when they perform mathematical tasks? We are interested in uncovering the step-by-step processes by which they arrive at solutions to mathematical problems. We want to know what it means for them to think mathematically.

When research provides a clearer picture of children's learning and thinking processes, the proper role of drill and practice should no longer be in doubt. But even so, a very practical question will remain—how to remove from the drill and practice its negative connotations. Should psychology and education manage to narrow the range of applicability of drill, to systematize guidelines for its organization, and to design maximally effective practice sequences, the situation will

be already be much improved. But all the artfulness a teacher or instructional designer can muster will still be needed to make practice interesting and self-motivating. Drill and practice may succeed only to the extent that it fills obvious needs, is presented in palatable formats, and can be convincingly related to the broader subject matter of mathematics.

SUMMARY

We have presented an overview of the psychological research surrounding the use of drill and practice to build arithmetic skills. The associationist theory of E. L. Thorndike, applied to the classroom, has been used to justify drill as a means of forming and strengthening the stimulus-response bonds that are viewed as constituting the subject matter of arithmetic. Although increases in computational speed and accuracy appear to accompany most drill, various arguments have been advanced against using it as a principal method of instruction. The main objection is that drill cannot develop quantitative thinking because it treats mathematics as a collection of isolated bonds rather than an integrated set of patterns and principles. In contrast to drill methods, "meaningful" instruction in the number concepts underlying computations has the virtue of acknowledging differences in the level of understanding possessed by children and adults. It also appears to enhance the generalizability of arithmetic skills.

Various attempts have been made over the years to specify the sequence and organization of drill and practice in computation. Some have sought to ensure proper ordering and amounts of drill based on determinations of relative difficulty of number facts. Others have isolated structural variables whose summed strength predicts performance on word problems and thus serves as a means of ordering those problems according to difficulty. Studies comparing isolated and mixed forms of drill suggest that concentrated practice is best used in close proximity to the introduction of new facts or procedures, and small amounts of practice at frequent intervals are best for maintaining facts and procedures once they are well learned. Computer-assisted drill programs represent an attempt to optimize the effectiveness of practice by adjusting the timing and difficulty level to individual students. Supplementary drill in this individualized mode was shown to improve children's performance on standardized achievement tests but not more than supplementary paper-and-pencil drill.

Recent information-processing theories of human memory point to a possible theoretical justification for drill and practice—the development of automaticity. The capacity to respond automatically to certain components of complex computations, such as number facts and simple algorithms, may reduce the processing load of the human memory system and thus contribute to its efficient functioning. More precise indications of the value and effects of drill and practice are necessary in order to define its proper role in mathematics instruction.

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